

Dechant, Pierre-Philippe ORCID:

<https://orcid.org/0000-0002-4694-4010> (2014) Affinne symmetry principles for non-crystallographic systems & applications to viruses/carbon onions. In: Mathematics Seminar, 16th December 2014, Doppler Institute, Technical University, Prague. (Unpublished)

Downloaded from: <http://ray.yorks.ac.uk/id/eprint/4024/>

Research at York St John (RaY) is an institutional repository. It supports the principles of open access by making the research outputs of the University available in digital form. Copyright of the items stored in RaY reside with the authors and/or other copyright owners. Users may access full text items free of charge, and may download a copy for private study or non-commercial research. For further reuse terms, see licence terms governing individual outputs. [Institutional Repository Policy Statement](#)

# RaY

Research at the University of York St John

For more information please contact RaY at [ray@yorks.ac.uk](mailto:ray@yorks.ac.uk)



THE UNIVERSITY *of York*



# Affine symmetry principles for non-crystallographic systems & applications to viruses/carbon onions

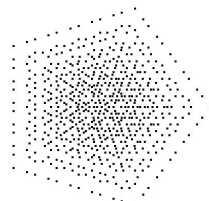
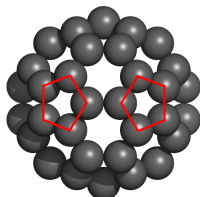
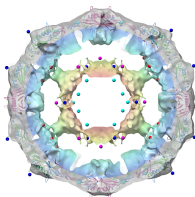
Pierre-Philippe Dechant

Mathematics Department, University of York  
Work with [Reidun Twarock](#) (York) and [Céline Boehm](#) (Durham)

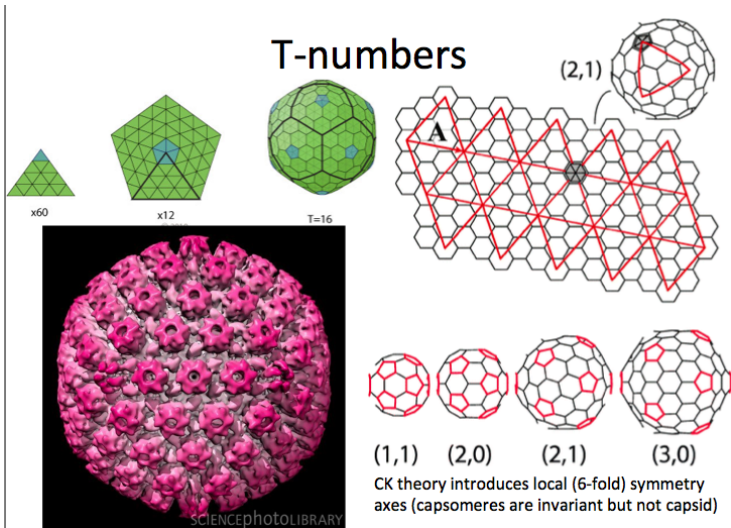
Doppler Institute, Technical University, Prague – December 16,  
2014

# Motivation: Viruses

- Geometry of **polyhedra** described by **Coxeter** groups
- Viruses have to be 'economical' with their **genes**
- Encode **structure** modulo **symmetry**
- **Largest discrete symmetry of space** is the **icosahedral** group
- Many other 'maximally symmetric' objects in nature are also icosahedral: **Fullerenes** & **Quasicrystals**
- But: viruses are not just polyhedral – they have **radial structure**. **Affine extensions** give **translations**

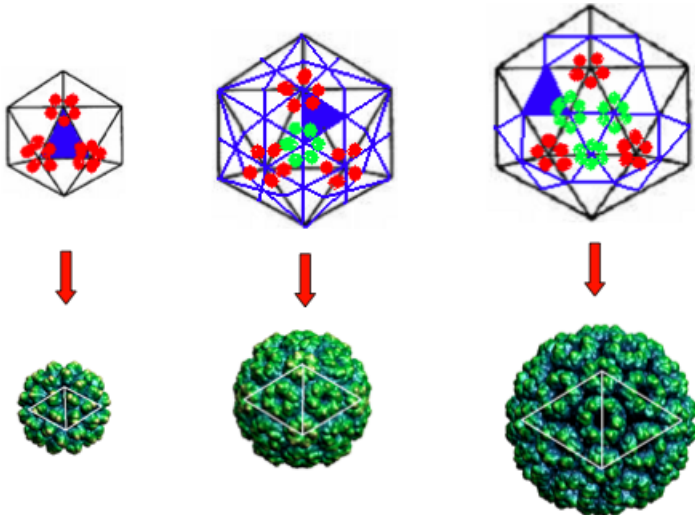


# Viruses: Caspar-Klug triangulations

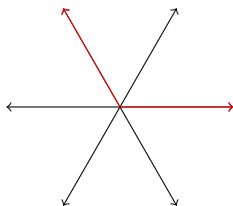




# Viruses: Caspar-Klug triangulations



# Root systems – $A_2$

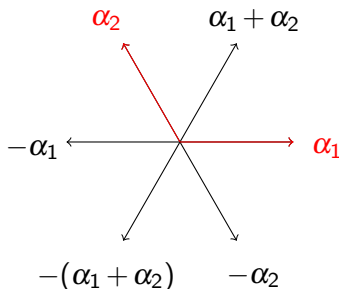


**Root system**  $\Phi$ : set of vectors  $\alpha$  such that

$$1. \quad \Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$$

$$2. \quad s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$$

# Root systems – $A_2$



**Root system**  $\Phi$ : set of vectors  $\alpha$  such that

- $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$

- $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

**Simple roots**: express every element of  $\Phi$  via a  **$\mathbb{Z}$ -linear combination** (with coefficients of the same sign).

# Cartan Matrices

Cartan matrix of  $\alpha_i$ s is

$$A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2 \frac{|\alpha_j|}{|\alpha_i|} \cos \theta_{ij}$$

# Cartan Matrices

Cartan matrix of  $\alpha_i$ s is  $A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2 \frac{|\alpha_j|}{|\alpha_i|} \cos \theta_{ij}$

angles  $\cos^2 \theta_{ij} = \frac{1}{4} A_{ij} A_{ji}$  lengths  $l_j^2 = \frac{A_{ij}}{A_{ji}} l_i^2$

$$A_{ii} = 2 \quad A_{ij} \in \mathbb{Z}^{\leq 0} \quad A_{ij} = 0 \Leftrightarrow A_{ji} = 0.$$

$$A_2: A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

# Cartan Matrices

Cartan matrix of  $\alpha_i$ s is 
$$A_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} = 2 \frac{|\alpha_j|}{|\alpha_i|} \cos \theta_{ij}$$

angles  $\cos^2 \theta_{ij} = \frac{1}{4} A_{ij} A_{ji}$  lengths  $l_j^2 = \frac{A_{ij}}{A_{ji}} l_i^2$

$$A_{ii} = 2 \quad A_{ij} \in \mathbb{Z}^{\leq 0} \quad A_{ij} = 0 \Leftrightarrow A_{ji} = 0.$$

$$A_2: A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

**Coxeter-Dynkin diagrams:** node = simple root, no link = roots orthogonal, simple link = roots at  $\frac{\pi}{3}$ , link with label  $m$  = angle  $\frac{\pi}{m}$ .

$$A_2 \circ \text{---} \circ \quad H_2 \circ \text{---}^5 \circ \quad I_2(n) \circ \text{---}^n \circ$$

# Coxeter groups

A **Coxeter group** is a group generated by some **involutive generators**  $s_i, s_j \in S$  subject to relations of the form  $(s_i s_j)^{m_{ij}} = 1$  with  $m_{ij} = m_{ji} \geq 2$  for  $i \neq j$ .

# Coxeter groups

A **Coxeter group** is a group generated by some **involutive generators**  $s_i, s_j \in S$  subject to relations of the form  $(s_i s_j)^{m_{ij}} = 1$  with  $m_{ij} = m_{ji} \geq 2$  for  $i \neq j$ .

The **finite** Coxeter groups have a **geometric representation** where the involutions are realised as **reflections** at **hyperplanes through the origin** in a Euclidean vector space  $\mathcal{E}$ . In particular, let  $(\cdot|\cdot)$  denote the inner product in  $\mathcal{E}$ , and  $v, \alpha \in \mathcal{E}$ .

The **generator**  $s_\alpha$  corresponds to the **reflection**

$$s_\alpha : v \rightarrow s_\alpha(v) = v - 2 \frac{(v|\alpha)}{(\alpha|\alpha)} \alpha$$

at a hyperplane perpendicular to the **root vector**  $\alpha$ .

The action of the **Coxeter group** is to permute these **root vectors**.



# Affine extensions

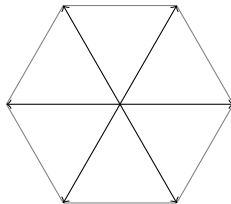
An **affine Coxeter group** is the extension of a Coxeter group by an **affine reflection in a hyperplane not containing the origin**  $s_{\alpha_0}^{aff}$  whose geometric action is given by

$$s_{\alpha_0}^{aff} v = \alpha_0 + v - \frac{2(\alpha_0 | v)}{(\alpha_0 | \alpha_0)} \alpha_0$$

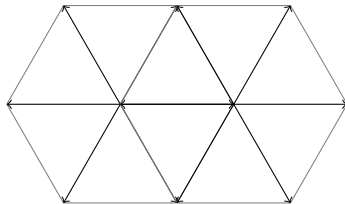
**Non-distance preserving:** includes the **translation generator**

$$T v = v + \alpha_0 = s_{\alpha_0}^{aff} s_{\alpha_0} v$$

# Affine extensions – $A_2$

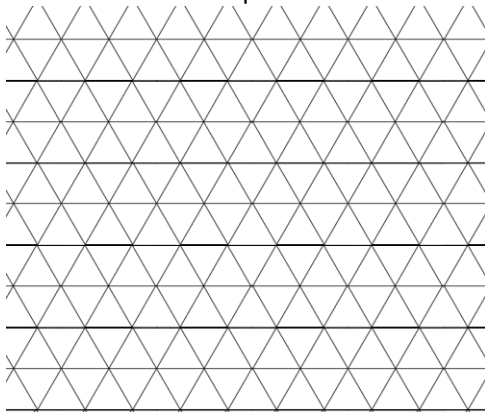


# Affine extensions – $A_2$

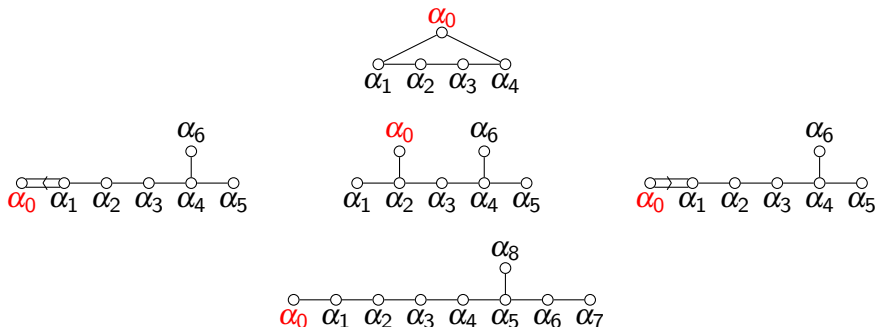


# Affine extensions – $A_2$

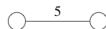
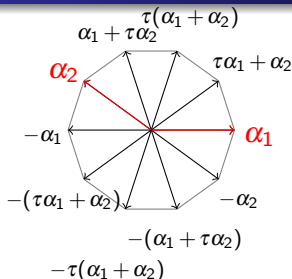
Affine extensions of crystallographic Coxeter groups lead to a **tessellation** of the plane and a **lattice**.



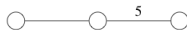
# Affine extensions of crystallographic groups $A_4$ , $D_6$ and $E_8$



# Non-crystallographic Coxeter groups $H_2 \subset H_3 \subset H_4$



$$A = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$H_2 \subset H_3 \subset H_4$ : 10, 120, 14,400 elements, the only Coxeter groups that generate **rotational symmetries of order 5** linear combinations now in the **extended integer ring**

$$\mathbb{Z}[\tau] = \{a + \tau b \mid a, b \in \mathbb{Z}\} \quad \text{golden ratio}$$

$$\tau = \frac{1}{2}(1 + \sqrt{5}) = 2 \cos \frac{\pi}{5}$$

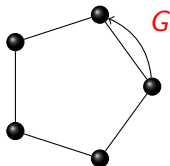
$$x^2 = x + 1$$

$$\tau' = \sigma = \frac{1}{2}(1 - \sqrt{5}) = 2 \cos \frac{2\pi}{5}$$

$$\tau + \sigma = 1, \tau\sigma = -1$$

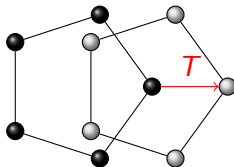
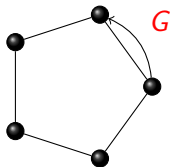
# Affine extensions of non-crystallographic root systems

Unit translation along a vertex of a unit pentagon



# Affine extensions of non-crystallographic root systems

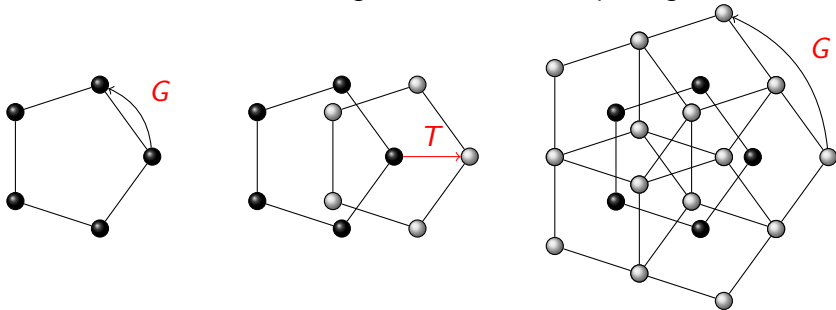
Unit translation along a vertex of a unit pentagon





# Affine extensions of non-crystallographic root systems

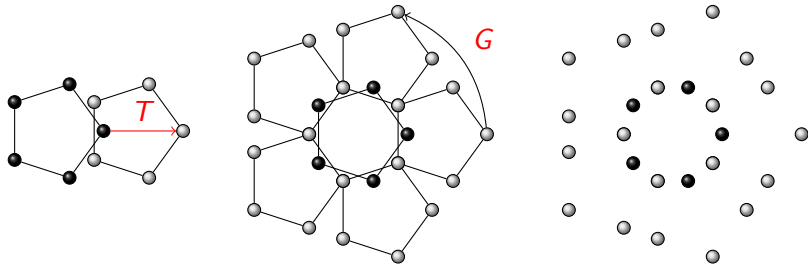
Unit translation along a vertex of a unit pentagon



A **random** translation would give 5 secondary pentagons, i.e. 25 points. Here we have **degeneracies** due to 'coinciding points'.

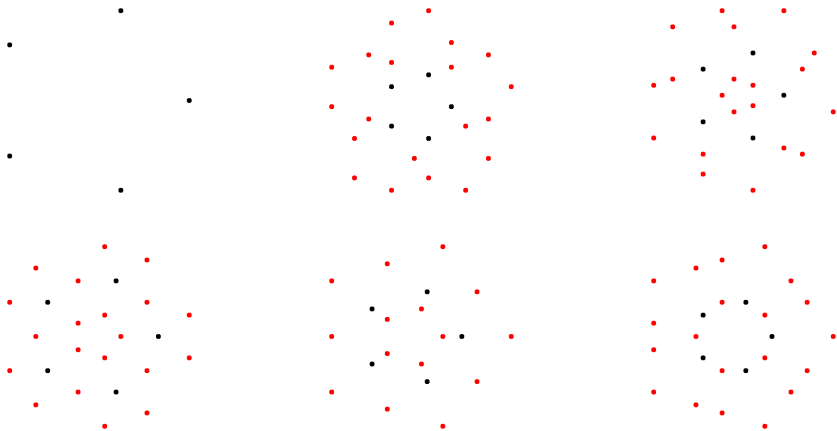
# Affine extensions of non-crystallographic root systems

Translation of length  $\tau = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$  (golden ratio)



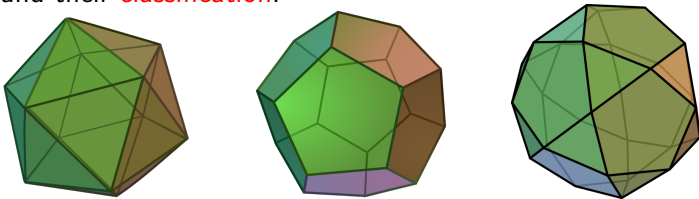
Looks like a **virus** or **carbon onion**

# More Blueprints

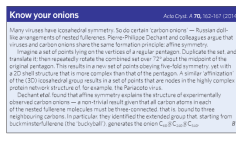
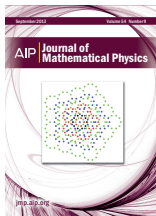


# Extend icosahedral group with distinguished translations

- Radial layers are **simultaneously constrained** by affine symmetry
- **Affine extensions** of the icosahedral group (giving translations) and their **classification**.



# Applications of affine extensions of non-crystallographic root systems



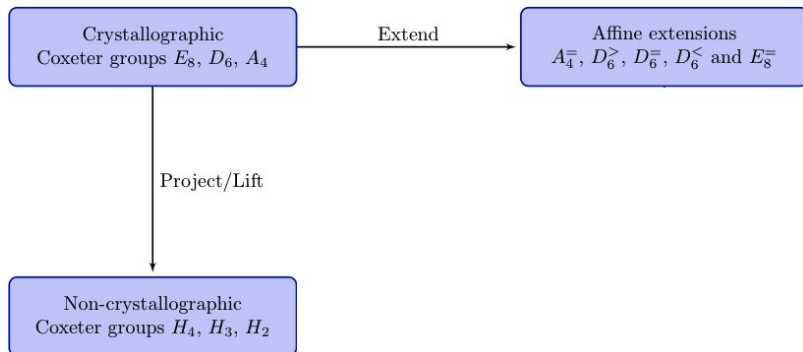
well-known effect for photons, and it turns out to hold for other quantum particles too. James Fickens and colleagues have performed the Hong-Ou-Mandel quantum interference experiment using plasmons, which are quantized surface plasma waves. Pairs of photons are fed into a specially designed photonic waveguide that mixes the paths of the light-excited surface plasmons in the same way as a beam splitter. The outcome is converted back into photons and measured by two detectors. As in the purely photonic case, the characteristic dip in coincidence rate is there, showing that the photons remain indistinguishable when they are converted into plasmons and interfere. 10

Written by May Chou, Kilo Georgescu, Albert Polner, Bert Voelckel and Alan Wong

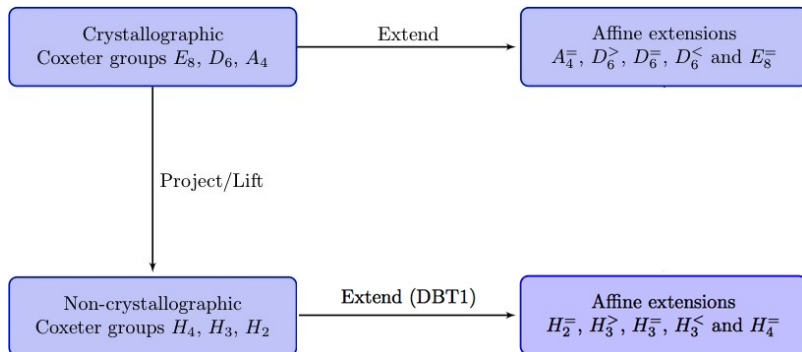
NATURE PHYSICS • VOL 10 | APRIL 2014 • www.nature.com/naturephysics

There are interesting applications to **quasicrystals**, **viruses** or **carbon onions** later, concentrate on the **mathematical** aspects for now

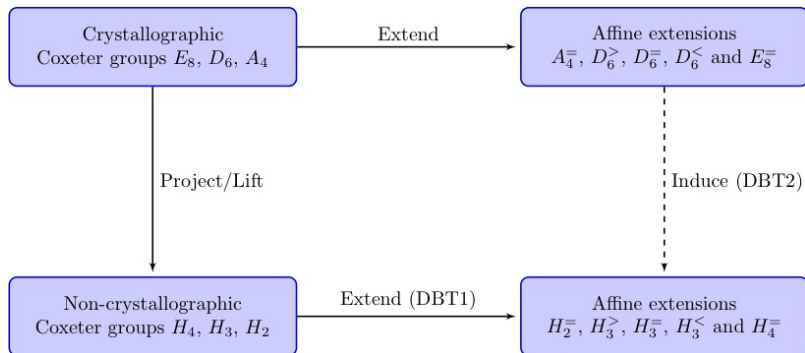
# Road Map



# Road Map



# Road Map





- 1 Affine extensions
  - Direct extensions
  - Induced extensions
- 2 Applications
  - Virus Structure
  - Fullerenes and Carbon onions
- 3 A 3D spinorial view of 4D exceptional phenomena
- 4 Conclusions

# Kac-Moody approach

Can recover these directly at the Cartan matrix level:

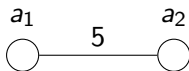
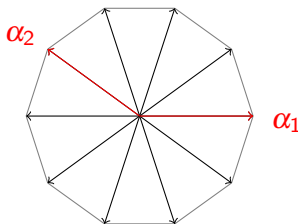
**Kac-Moody-type affine extension**  $A^{aff}$  of a Cartan matrix is an extension of the Cartan matrix  $A$  of a Coxeter group by further **rows**  $\underline{v}$  and **columns**  $\underline{w}$  such that:

$$A^{aff} = \begin{pmatrix} 2 & \underline{v}^T \\ \underline{w} & A \end{pmatrix} \quad \boxed{A_{ii}^{aff} = 2} \quad \boxed{A_{ij}^{aff} \in \mathbb{Z}[\cdot]}$$

$$\boxed{A_{ij}^{aff} \leq 0} \quad \text{moreover,} \quad \boxed{A_{ij}^{aff} = 0 \Leftrightarrow A_{ji}^{aff} = 0}$$

**determinant constraint**  $\boxed{\det A^{aff} = 0}$

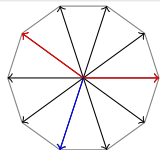
# Kac-Moody approach to $H_2$

 $\overset{5}{\circ}-\circ$ 


$$\alpha_1 = (1, 0), \quad \alpha_2 = \frac{1}{2}(-\tau, \sqrt{3-\tau})$$

$$A = \begin{pmatrix} 2 & \cdot & \cdot \\ \cdot & 2 & -\tau \\ \cdot & -\tau & 2 \end{pmatrix}$$

## Extension along the highest root

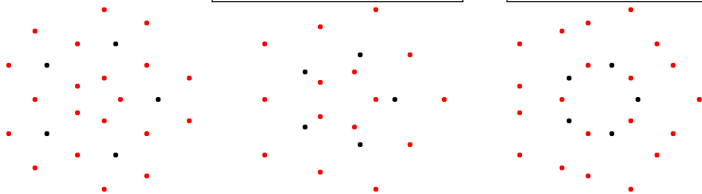


$$A = \begin{pmatrix} 2 & x & x \\ y & 2 & -\tau \\ y & -\tau & 2 \end{pmatrix}$$

$$xy = 2 - \tau = \sigma^2$$

symmetric  $x = y = \sigma = 1 - \tau$  recovers  $H_2^{aff}$  from Twarock et al

new asymmetric e.g.  $(x, y) = (\tau - 2, -1)$  or  $(x, y) = (-1, \tau - 2)$



Write  $x = (a + \tau b)$  and  $y = (c + \tau d)$  with  $a, b, c, d \in \mathbb{Z}$ , i.e.  $H_2^{aff}$  is  $(a, b; c, d) = (1, -1; 1, -1)$ .

# Fibonacci scaling

The (non-trivial) **units** in  $\mathbb{Z}[\tau]$  are  $\tau^k$ ,  $k \in \mathbb{Z}$

Can **generate all solutions** to the determinant constraint  $\boxed{xy = \sigma^2}$   
by

**scaling**  $\boxed{x \rightarrow \tau^{-k}x, y \rightarrow \tau^k y}$ :  $xy$  invariant (giving the **angle**),

but different **lengths**  $\sqrt{\frac{x}{y}} \rightarrow \sqrt{\frac{x}{y}}\tau^{-k}$

**Fibonacci scaling**

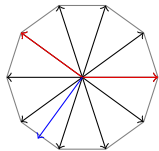
$(a, b; c, d) \rightarrow (b, a + b; d - c, c)$  for multiplication by  $(\tau, \tau^{-1})$  and

$(a, b; c, d) \rightarrow (b - a, a; d, c + d)$  for multiplication by  $(\tau^{-1}, \tau)$

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

**Swapping**  $\boxed{x \leftrightarrow y}$  generates another solution, but here symmetric

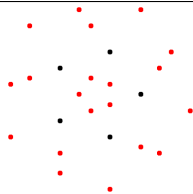
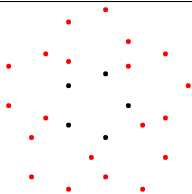
# Extension along a bisector



$$A = \begin{pmatrix} 2 & x & 0 \\ y & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}$$

$$xy = 3 - \tau$$

$$(x, y) = (\tau - 3, -1) \text{ or } (x, y) = (-1, \tau - 3)$$



# Extension along the highest root – two-fold axis $T_2$

$$\alpha_1 = (0, 1, 0), \quad \alpha_2 = -\frac{1}{2}(-\sigma, 1, \tau), \quad \alpha_3 = (0, 0, 1)$$

$$T_2 = (1, 0, 0)$$

$$A = \begin{pmatrix} 2 & 0 & \textcolor{red}{x} & 0 \\ 0 & 2 & -1 & 0 \\ \textcolor{red}{y} & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$$\textcolor{red}{xy} = \sigma^2 = 2 - \tau$$

Same solution as in the previous case of  $H_2$ .

# Extension along a three-fold axis $T_3$

$$\alpha_1 = (0, 1, 0), \alpha_2 = -\frac{1}{2}(-\sigma, 1, \tau), \alpha_3 = (0, 0, 1)$$

$$T_3 = (\tau, 0, \sigma)$$

$$A = \begin{pmatrix} 2 & 0 & 0 & x \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ y & 0 & -\tau & 2 \end{pmatrix}$$

$$xy = \frac{4}{3}\sigma^2$$

No longer  $\mathbb{Z}[\tau]$ -valued, and hence **solutions do not exist in  $\mathbb{Z}[\tau]$** .  
 What now? Allow  $\mathbb{Q}[\tau]$ ? Write  $x = \gamma(a + \tau b)$  and  $y = \delta(c + \tau d)$   
 with  $a, b, c, d \in \mathbb{Z}$  and  $\gamma, \delta \in \mathbb{Q}$ . Need  $\gamma\delta = \frac{4}{3}$ , then can recycle  
 integer solution



# Extension along a five-fold axis $T_5$

$$\alpha_1 = (0, 1, 0), \quad \alpha_2 = -\frac{1}{2}(-\sigma, 1, \tau), \quad \alpha_3 = (0, 0, 1)$$

$$T_5 = (\tau, -1, 0)$$

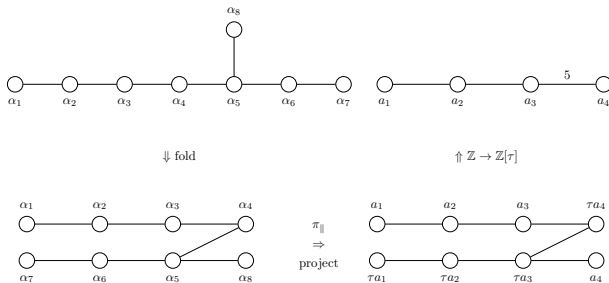
$$A = \begin{pmatrix} 2 & x & 0 & 0 \\ y & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$$xy = \frac{4}{5}(3 - \tau)$$

Same solution (two series) as before in the case of  $H_2$ , but this time with the additional degree of freedom.

- 1 Affine extensions
  - Direct extensions
  - Induced extensions
- 2 Applications
  - Virus Structure
  - Fullerenes and Carbon onions
- 3 A 3D spinorial view of 4D exceptional phenomena
- 4 Conclusions

# Projection and Diagram Foldings



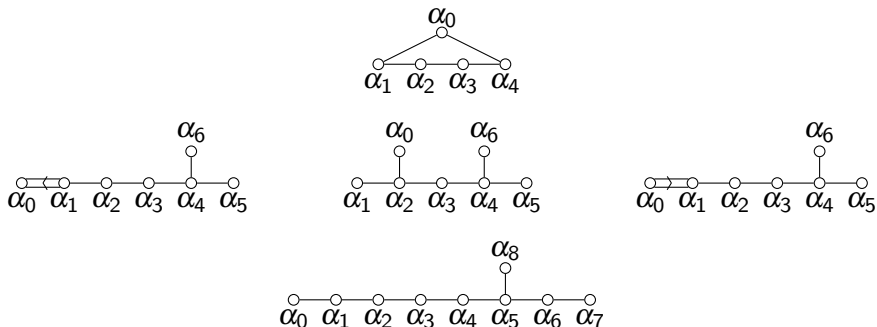
$$s_{\beta_1} = s_{\alpha_1} s_{\alpha_7}, \quad s_{\beta_2} = s_{\alpha_2} s_{\alpha_6}, \quad s_{\beta_3} = s_{\alpha_3} s_{\alpha_5}, \quad s_{\beta_4} = s_{\alpha_4} s_{\alpha_8} \Rightarrow H_4$$

$E_8$  has two  $H_4$ -invariant subspaces – blockdiagonal form

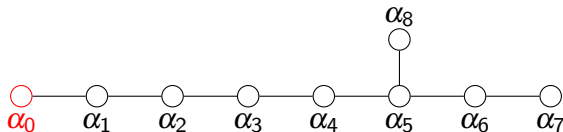
$D_6$  has two  $H_3$ -invariant subspaces

$A_4$  has two  $H_2$ -invariant subspaces

# Recap: Affine extensions of crystallographic groups



# Affine extensions – $E_8^-$

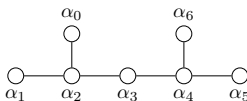


$$-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$$

AKA  $E_8^+$  and along with  $E_8^{++}$  and  $E_8^{+++}$  thought to be the underlying symmetry of **String and M-theory**

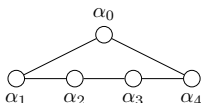
Also interesting from a pure mathematics point of view:  **$E_8$  lattice**, **McKay correspondence** and **Monstrous Moonshine**.

# Affine extensions – simply-laced $D_6^=$ , $A_4^=$



$$A(D_6^=) = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

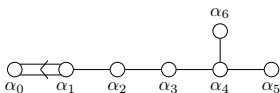
$$-\alpha_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$$



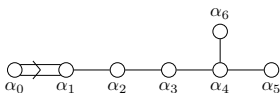
$$A(A_4^=) = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$-\alpha_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

# Affine extensions – $D_6^<$ and $D_6^>$



$$A(D_6^<) = \begin{pmatrix} 2 & -2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$



$$A(D_6^>) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$-\alpha_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \frac{1}{2}\alpha_5 + \frac{1}{2}\alpha_6$$

$$-\alpha_0 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$$

# Induced affine roots: $H_4^-$ from $E_8^-$

$$-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$$

$$-a_0 = \pi_{||}(-\alpha_0) = 2(1+\tau)a_1 + (3+4\tau)a_2 + 2(2+3\tau)a_3 + (3+5\tau)a_4$$

$$(a_1|a_2) = -\frac{1}{2}, (a_2|a_3) = -\frac{1}{2}, (a_3|a_4) = -\frac{\tau}{2},$$

$$A(H_4^-) := \begin{pmatrix} 2 & \tau-2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -\tau \\ 0 & 0 & 0 & -\tau & 2 \end{pmatrix}$$

induced affine root of lengths  $\tau$  and  $1/\tau$  along the highest root  $\alpha_H = (1, 0, 0, 0)$  of  $H_4$



# Induced affine extensions: $H_i^-$ from $A_4^-$ , $D_6^-$ and $E_8^-$

affine extensions of lengths  $\tau$  and  $1/\tau$  along the highest root  $\alpha_H$  of

$$A(H_4^-) := \begin{matrix} & H_i \\ \begin{pmatrix} 2 & \tau-2 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -\tau \\ 0 & 0 & 0 & -\tau & 2 \end{pmatrix} \end{matrix}$$

$$A(H_3^-) := \begin{pmatrix} 2 & 0 & \tau-2 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$$A(H_2^-) := \begin{pmatrix} 2 & \tau-2 & \tau-2 \\ -1 & 2 & -\tau \\ -1 & -\tau & 2 \end{pmatrix}$$

# Induced affine extensions: three $H_3^+$ from $D_6^+$

$$A(H_3^=) := \begin{pmatrix} 2 & 0 & \tau-2 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

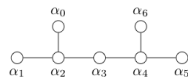
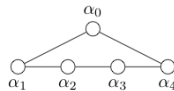
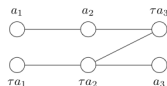
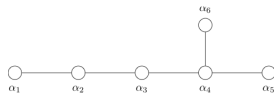
$$A(H_3^<) := \begin{pmatrix} 2 & \frac{4}{5}(\tau-3) & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

$$A(H_3^>) := \begin{pmatrix} 2 & \frac{2}{5}(\tau-3) & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}$$

# Comparison with DBT1

- $H_i^{aff}$  was the **symmetric special case** of the **Fibonacci 'family' of solutions**
- $H_i^=$  induced by projection of the affine extensions  $E_8^=$ ,  $D_6^=$ ,  $A_4^=$  is the **'first asymmetric case'**
- Achieved by **scaling** the symmetric solution of  $H_i^{aff}$  by  $(\tau, \tau^{-1})$
- Projection from  $D_6^<$  and  $D_6^>$  give extensions along **5-fold axes** of icosahedral symmetry, from  $D_6^=$  along **2-fold axes**
- These are exactly what we were looking for for icosahedral applications!

# Invariance under Dynkin diagram automorphisms



$$-\alpha_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

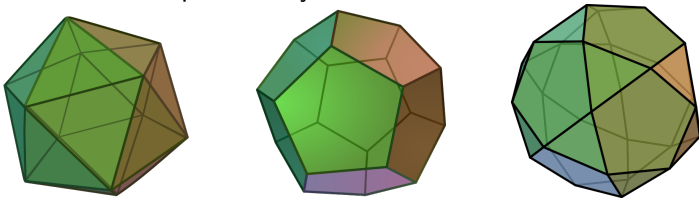
$$-\alpha_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$$

$$-\alpha_0 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$$

- 1 Affine extensions
  - Direct extensions
  - Induced extensions
- 2 Applications
  - Virus Structure
  - Fullerenes and Carbon onions
- 3 A 3D spinorial view of 4D exceptional phenomena
- 4 Conclusions

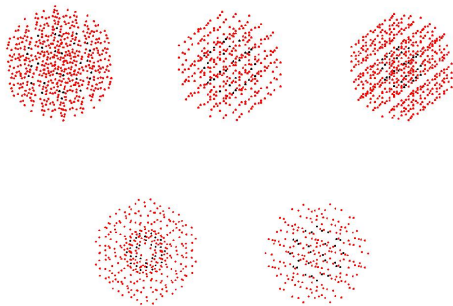
## Extend icosahedral group with distinguished translations

- Radial layers are **simultaneously constrained** by affine symmetry
- Works very well in practice: **finite library of blueprints**
- **Select** blueprint from the **outer shape** (capsid)
- Can **predict inner structure** (nucleic acid distribution) of the virus from the point array



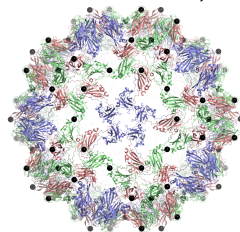
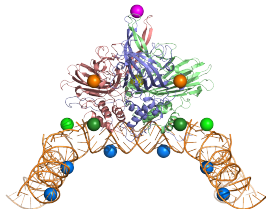
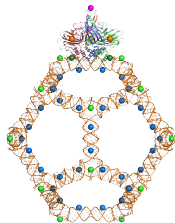
**Affine extensions** of the icosahedral group (giving translations) and their **classification**.

# What's the point?



# Use in Mathematical Virology

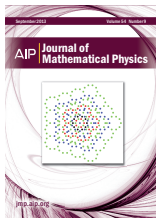
- Suffice to say **point arrays work very exceedingly well** in practice. Two papers on the mathematical (Coxeter) aspects.
- **Implemented computational problem in Clifford** – some **very interesting mathematics** comes out as well (see paper 'Platonic solids generate their 4-dimensional analogues').





# Use in Mathematical Virology

- Suffice to say **point arrays work very exceedingly well** in practice.
- **Implemented computational problem in Clifford algebra** – some **very interesting mathematics** comes out as well (see paper ‘Platonic solids generate their 4-dimensional analogues’).



## Know your onions

Acta Cryst. A 70, 165-167 (2014)

Many viruses have icosahedral symmetry. So do certain ‘carbon onions’ — Russian doll-like arrangements of nested fullerenes. Pierre-Philippe Dechant and colleagues argue that viruses and carbon onions share the same formation principle: affine symmetry. Imagine a set of points lying on the vertices of a regular pentagon. Duplicate the set, and translate it, then repeatedly rotate the combined set over  $12^\circ$  about the midpoint of the original pentagon. This results in a new set of points obeying five-fold symmetry, yet with a 2D shell structure that is more complex than that of the pentagon. A similar affinitization of the 3D icosahedral group results in a set of points that are nodes in the highly complex protein network structure of, for example, the Pentateco virus. Dechant et al. found that affine symmetry explains the structure of experimentally observed carbon onions — a non-trivial result given that all carbon atoms in each of the nested fullerene molecules must be three-connected, that is, bound to three neighbouring carbons. In particular they identified the extended group that, starting from buckminsterfullerene (the ‘buckyball’), generates the onion  $C_{60}@C_{60}@C_{60}$ .

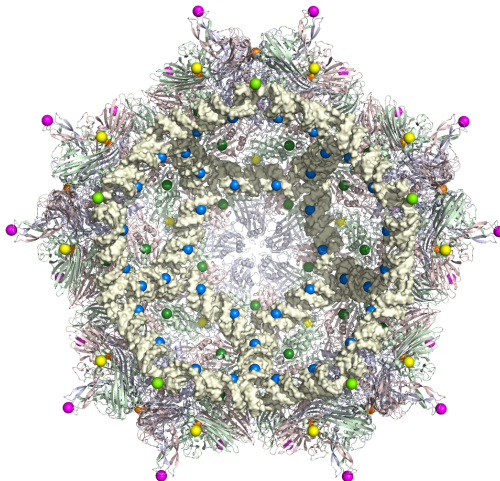
well-known effect for photons, and it turns out to hold for other quantum particles too. James Fickner and colleagues have performed the Hong–Ou–Mandel quantum interference experiment using plasmons, which are quantized surface plasma waves. Pairs of photons are fed into a specially designed plasmonic waveguide that mixes the paths of the light-scattered surface plasmons in the same way as a beam splitter. The outcome is converted back into photons and measured by two detectors. As in the purely photonic case, the characteristic dip in coincidence rate is there, showing that the photons remain indistinguishable when they are converted into plasmons and interfere.

Written by Amy Chu, Miki Georgiou, Abigail Knappe, Bert Voelckers and Alison Wright

NATURE PHYSICS | VOL 10 | APRIL 2014 | www.nature.com/naturephysics

244

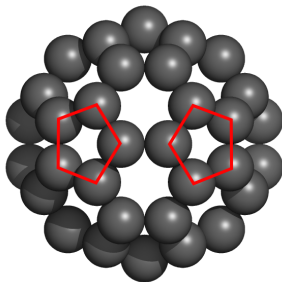
# Use in Mathematical Virology



- 1 Affine extensions
  - Direct extensions
  - Induced extensions
- 2 Applications
  - Virus Structure
  - Fullerenes and Carbon onions
- 3 A 3D spinorial view of 4D exceptional phenomena
- 4 Conclusions

# Constraints of carbon chemistry

- Relevant carbon bonding here is **trivalent**
- **Bond lengths and angles** need to be pretty **uniform**
- For example, the well-known **football**-shaped Buckyball  $C_{60}$



# Strategy

- Extend icosahedral shapes with a **translation** and take orbit under the compact group
- Select **outer shells** that are **three-coordinated** and uniform enough
- For the usual **icosahedron**, **dodecahedron**, **icosidodecahedron** find few not very interesting possibilities
- For  **$C_{60}$**  and  **$C_{80}$**  start, get a **unique** extension that exactly give the known **carbon onions**  $C_{60} - C_{240} - C_{540}$  and  $C_{80} - C_{180} - C_{320}$

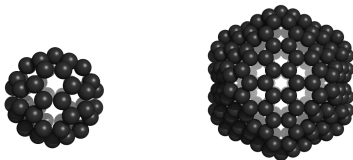
# Fullerene cages derived from $C_{60}$

- Extend idea of affine symmetry to other objects in nature: icosahedral **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach starting with  $C_{60}$ : **carbon onion** ( $C_{60} - C_{240} - C_{540}$ )



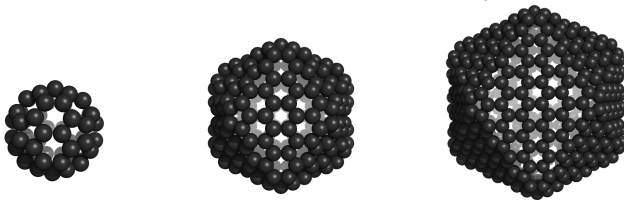
# Fullerene cages derived from $C_{60}$

- Extend idea of affine symmetry to other objects in nature: icosahedral **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach starting with  $C_{60}$ : **carbon onion** ( $C_{60} - C_{240} - C_{540}$ )



# Fullerene cages derived from $C_{60}$

- Extend idea of affine symmetry to other objects in nature: icosahedral **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach starting with  $C_{60}$ : **carbon onion** ( $C_{60} - C_{240} - C_{540}$ )





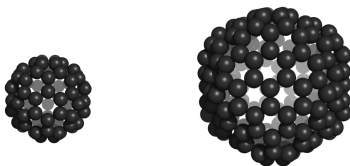
# Fullerene cages derived from $C_{80}$

- Extend idea of affine symmetry to other objects in nature: icosahedral **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach starting with  $C_{80}$ : **carbon onion** ( $C_{80} - C_{180} - C_{320}$ )



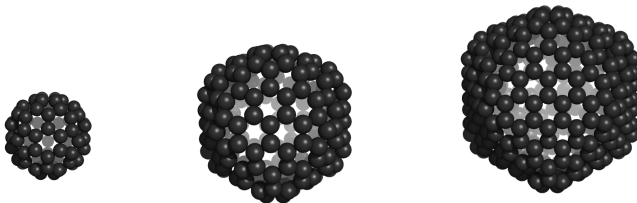
# Fullerene cages derived from $C_{80}$

- Extend idea of affine symmetry to other objects in nature: icosahedral **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach starting with  $C_{80}$ : **carbon onion** ( $C_{80} - C_{180} - C_{320}$ )



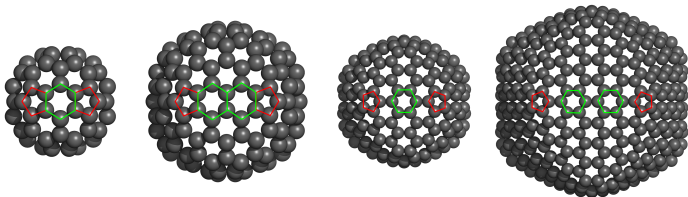
# Fullerene cages derived from $C_{80}$

- Extend idea of affine symmetry to other objects in nature: icosahedral **fullerenes**
- Recover different shells with icosahedral symmetry from affine approach starting with  $C_{80}$ : **carbon onion** ( $C_{80} - C_{180} - C_{320}$ )



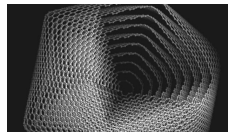
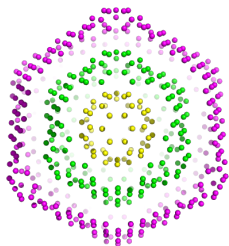
# Growth of shells by a hexamer at a time

- Hence, for  $C_{60}$  and  $C_{80}$  start, get a **unique** extension that exactly give the known **carbon onions**  $C_{60} - C_{240} - C_{540}$  and  $C_{80} - C_{180} - C_{320}$  by inserting an **additional hexamer** at each step



# Viruses and fullerenes – symmetry as a common thread?

- Get nested arrangements like Russian dolls: **carbon onions** (e.g. June: Nature 510, 250253)
- Potential to extend to **other known carbon onions** with different start configuration, chirality etc



## References (collaborations)

- Novel Kac-Moody-type affine extensions of non-crystallographic Coxeter groups with Twarock/Bøhm J. Phys. A: Math. Theor. 45 285202 (2012)
- Affine extensions of non-crystallographic Coxeter groups induced by projection with Twarock/Bøhm Journal of Mathematical Physics 54 093508 (2013), [Cover article September](#)
- Viruses and Fullerenes – Symmetry as a Common Thread? with Twarock/Wardman/Keef Acta Crystallographica A 70 (2). pp. 162-167 (2014), [Cover article March](#), [Nature Physics Research Highlight](#)

## References (single-author)

- Clifford algebra unveils a surprising geometric significance of quaternionic root systems of Coxeter groups  
Advances in Applied Clifford Algebras 23 (2). pp. 301-321 (2013)
- A Clifford algebraic framework for Coxeter group theoretic computations (Conference Prize at AGACSE 2012)  
Advances in Applied Clifford Algebras 24 (1). pp. 89-108 (2014)
- Rank-3 root systems induce root systems of rank 4 via a new Clifford spinor construction Journal of Physics (2015) – accepted today!
- Platonic Solids generate their 4-dimensional analogues  
Acta Cryst. A69 (2013)

# 3D vs 4D

- Have  $A_n$ ,  $B_n$  and  $D_n$  families of root systems in **any dimension**
- In **3D**, have  $H_3$  as an **accident** (icosahedron and dodecahedron)
- In **4D**, have  $F_4$  and  $H_4$  (and in some sense  $D_4$ ) as accidents
- These 4D accidents have **unusual automorphism groups**
- Can **induce** all of these from the 3D cases, show they are **root systems** and explain their **automorphism groups**



# Clifford Algebra and orthogonal transformations

- Form an algebra using the **Geometric Product** for two vectors

$$ab \equiv a \cdot b + a \wedge b$$

- Inner product** is symmetric  $a \cdot b = \frac{1}{2}(ab + ba)$
- Reflecting  $a$  in  $b$  is given by  $a' = a - 2(a \cdot b)b = -bab$  ( $b$  and  $-b$  **doubly cover** the same reflection)
- Via **Cartan-Dieudonné** theorem any orthogonal transformation can be written as **successive reflections**

$$x' = \pm n_1 n_2 \dots n_k x n_k \dots n_2 n_1 = \pm A x \tilde{A}$$

# Clifford Algebra of 3D

- E.g. **Pauli algebra** in 3D (likewise for **Dirac algebra** in 4D) is

$$\begin{array}{cccc}
 \underbrace{\{1\}} & \underbrace{\{e_1, e_2, e_3\}} & \underbrace{\{e_1 e_2, e_2 e_3, e_3 e_1\}} & \underbrace{\{I \equiv e_1 e_2 e_3\}} \\
 1 \text{ scalar} & 3 \text{ vectors} & 3 \text{ bivectors} & 1 \text{ trivector}
 \end{array}$$

- We can form the elements of the Coxeter groups by **multiplying together root vectors** in this algebra  $\alpha_i \alpha_j \dots$
- In general get something with 8 components, here restrict to **even** products (rotations/spinors) with **four** components:

$$R = a_0 + a_1 e_2 e_3 + a_2 e_3 e_1 + a_3 e_1 e_2 \Rightarrow R \tilde{R} = a_0^2 + a_1^2 + a_2^2 + a_3^2$$

- So behaves as a **4D Euclidean** object – norm  
 $(R_1, R_2) = \frac{1}{2}(R_2 \tilde{R}_1 + R_1 \tilde{R}_2)$

# Induction Theorem – root systems

- Theorem: 3D spinor groups give 4D root systems.

# Induction Theorem – root systems

- Theorem: 3D spinor groups give 4D root systems.

- Check axioms:

1.  $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$

2.  $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$

# Induction Theorem – root systems

- Theorem: 3D spinor groups give 4D root systems.
- Check axioms:
  1.  $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \quad \forall \alpha \in \Phi$
  2.  $s_\alpha \Phi = \Phi \quad \forall \alpha \in \Phi$
- Proof: 1.  $R$  and  $-R$  are in a spinor group by construction (double cover of orthogonal transformations), 2. closure under reflections is guaranteed by the closure property of the spinor group (with a twist:  $-R_1 \tilde{R}_2 R_1$ )
- Induction Theorem: Every rank-3 root system induces a rank-4 root system (and thereby Coxeter groups)
- Counterexample: not every rank-4 root system is induced in this way

# Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the **Platonic Solids**:
- The 6 **reflections** in  $A_1 \times A_1 \times A_1$  generate 8 **spinors**.
- $\pm e_1, \pm e_2, \pm e_3$  give the 8 spinors  $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The **discrete spinor group** is isomorphic to the **quaternion** group  $Q$ .

# Spinors from reflections

- The 3D Coxeter groups that are symmetry groups of the **Platonic Solids**:
- The 6/12/18/30 **reflections** in  $A_1 \times A_1 \times A_1 / A_3 / B_3 / H_3$  generate 8/24/48/120 **spinors**.
- E.g.  $\pm e_1, \pm e_2, \pm e_3$  give the 8 spinors  $\pm 1, \pm e_1 e_2, \pm e_2 e_3, \pm e_3 e_1$
- The **discrete spinor group** is isomorphic to the **quaternion group**  $Q$  / **binary tetrahedral group**  $2T$  / **binary octahedral group**  $2O$  / **binary icosahedral group**  $2I$ ).

$A_1^3$	$A_3$	$B_3$	$H_3$
$A_1^4$	$D_4$	$F_4$	$H_4$

# Exceptional Root Systems

- The 16-cell, 24-cell, 24-cell and dual 24-cell and the 600-cell are in fact the root systems of  $A_1 \times A_1 \times A_1 \times A_1$ ,  $D_4$ ,  $F_4$  and  $H_4$
- Exceptional phenomena:  $D_4$  (triality, important in string theory),  $F_4$  (largest lattice symmetry in 4D),  $H_4$  (largest non-crystallographic symmetry)
- Exceptional  $D_4$  and  $F_4$  arise from series  $A_3$  and  $B_3$
- In fact, as we have seen one can strengthen this statement on inducing polytopes to a statement on inducing root systems



# Root systems in three and four dimensions

The **spinors** generated from the reflections contained in the respective **rank-3 Coxeter group** via the geometric product are realisations of the **binary polyhedral groups**  $Q$ ,  $2T$ ,  $2O$  and  $2I$ , which were known to generate (mostly exceptional) **rank-4 groups**, but **not known why**, and why the '**mysterious symmetries**'.

rank-3 group	diagram	binary	rank-4 group	diagram
$A_1 \times A_1 \times A_1$		$Q$	$A_1 \times A_1 \times A_1 \times A_1$	
$A_3$		$2T$	$D_4$	
$B_3$		$2O$	$F_4$	
$H_3$		$2I$	$H_4$	

# Induction Theorem – automorphism

- So induced **4D polytopes** are actually **root systems**.
- Clear why the **number of roots**  $|\Phi|$  is equal to  $|G|$ , the **order of the spinor group**
- Spinor group is trivially **closed** under **conjugation, left and right multiplication**. Results in **non-trivial symmetries** when viewed as a **polytope/root system**.
- Now explains **symmetry** of the polytopes/root system and thus the **order** of the rank-4 Coxeter group
- Theorem: The **automorphism group** of the induced root system contains **two factors** of the respective spinor group acting from the **left** and the **right**.

# Spinorial Symmetries of 4D Polytopes

## Spinorial symmetries

rank 3	$ \Phi $	$ W $	rank 4	$ \Phi $	Symmetry
$A_3$	12	24	$D_4$ 24-cell	24	$2 \cdot 24^2 = 576$
$B_3$	18	48	$F_4$ lattice	48	$48^2 = 2304$
$H_3$	30	120	$H_4$ 600-cell	120	$120^2 = 14400$
$A_1^3$	6	8	$A_1^4$ 16-cell	8	$3! \cdot 8^2 = 384$
$A_1 \oplus A_2$	8	12	$A_2 \oplus A_2$ prism	12	$12^2 = 144$
$A_1 \oplus H_2$	12	20	$H_2 \oplus H_2$ prism	20	$20^2 = 400$
$A_1 \oplus I_2(n)$	$n+2$	$2n$	$I_2(n) \oplus I_2(n)$	$2n$	$(2n)^2$

Similar for **Grand Antiprism** ( $H_4$  without  $H_2 \oplus H_2$ ) and **Snub 24-cell** ( $2I$  without  $2T$ ). Additional factors in the automorphism group come from **3D Dynkin diagram symmetries**!

## Some non-Platonic examples of spinorial symmetries

- **Grand Antiprism**: the 100 vertices achieved by subtracting 20 vertices of  $H_2 \oplus H_2$  from the 120 vertices of the  $H_4$  root system 600-cell – two separate orbits of  $H_2 \oplus H_2$
- This is a semi-regular polytope with automorphism symmetry  $\text{Aut}(H_2 \oplus H_2)$  of order  $400 = 20^2$
- Think of the  $H_2 \oplus H_2$  as coming from the **doubling procedure?** (Likewise for  $\text{Aut}(A_2 \oplus A_2)$  subgroup)
- **Snub 24-cell**:  $2T$  is a subgroup of  $2I$  so subtracting the 24 corresponding vertices of the 24-cell from the 600-cell, one gets a semiregular polytope with 96 vertices and automorphism group  $2T \times 2T$  of order  $576 = 24^2$ .

# Arnold's Trinities

Arnold's observation that many areas of real mathematics can be **complexified** and **quaternionified** resulting in theories with a similar structure.

- The **fundamental trinity** is thus  $(\mathbb{R}, \mathbb{C}, \mathbb{H})$
- The **projective spaces**  $(\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n)$
- The **spheres**  $(\mathbb{R}P^1 = S^1, \mathbb{C}P^2 = S^2, \mathbb{H}P^1 = S^4)$
- The **Möbius/Hopf bundles**  $(S^1 \rightarrow S^1, S^4 \rightarrow S^2, S^7 \rightarrow S^4)$
- The **Lie Algebras**  $(E_6, E_7, E_8)$
- The symmetries of the **Platonic Solids**  $(A_3, B_3, H_3)$
- The **4D groups**  $(D_4, F_4, H_4)$
- **New connections** via my **Clifford spinor construction** (see McKay correspondence)

# Platonic Trinities

- Arnold's connection between  $(A_3, B_3, H_3)$  and  $(D_4, F_4, H_4)$  is **very convoluted** and involves numerous other trinities at intermediate steps:
- **Decomposition of the projective plane** into Weyl chambers and Springer cones
- The **number of Weyl chambers** in each segment is  $24 = 2(1 + 3 + 3 + 5)$ ,  $48 = 2(1 + 5 + 7 + 11)$ ,  $120 = 2(1 + 11 + 19 + 29)$
- Notice this miraculously **matches the quasihomogeneous weights**  $((2, 4, 4, 6), (2, 6, 8, 12), (2, 12, 20, 30))$  of the Coxeter groups  $(D_4, F_4, H_4)$
- Believe the Clifford connection is **more direct**

# A unified framework for polyhedral groups

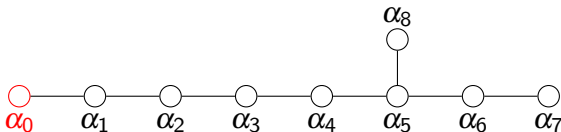
Group	Discrete subgroup	Action Mechanism
$SO(3)$	rotational (chiral)	$x \rightarrow \tilde{R}xR$
$O(3)$	reflection (full/Coxeter)	$x \rightarrow \pm \tilde{A}xA$
$Spin(3)$	binary	$(R_1, R_2) \rightarrow R_1 R_2$
$Pin(3)$	pinor	$(A_1, A_2) \rightarrow A_1 A_2$

- e.g. the **chiral icosahedral** group has 60 elements, encoded in Clifford by 120 spinors, which form the **binary icosahedral** group
- together with the **inversion/pseudoscalar**  $I$  this gives 60 rotations and 60 rotoinversions, i.e. the **full icosahedral** group  $H_3$  in 120 elements (with 240 pinors)
- all three are interesting groups, e.g. in **neutrino and flavour physics** for family symmetry model building

# Some Group Theory: chiral, full, binary, pin

- Easy enough to calculate **conjugacy classes** etc of pinors in Clifford algebra
- Chiral (**binary**) polyhedral groups have irreps
- tetrahedral (12/24):  $1, 1', 1'', 2_s, 2'_s, 2''_s, 3$
- octahedral (24/48):  $1, 1', 2, 2_s, 2'_s, 3, 3', 4_s$
- icosahedral (60/120):  $1, 2_s, 2'_s, 3, \bar{3}, 4, 4_s, 5, 6_s$
- Binary groups are **discrete subgroups of  $SU(2)$**  and all thus have a  $2_s$  spinor irrep
- Connection with the **McKay correspondence!**



Affine extensions –  $E_8^-$ 

$$-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$$

AKA  $E_8^+$  and along with  $E_8^{++}$  and  $E_8^{+++}$  thought to be the underlying symmetry of **String and M-theory**

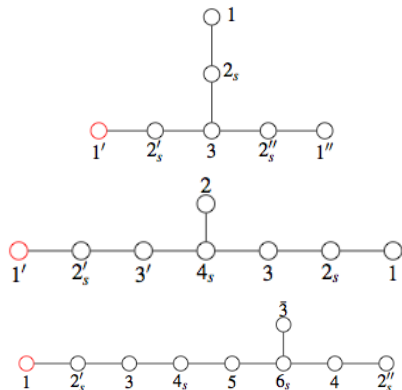
Also interesting from a pure mathematics point of view:  **$E_8$  lattice**, **McKay correspondence** and **Monstrous Moonshine**.

# The McKay Correspondence

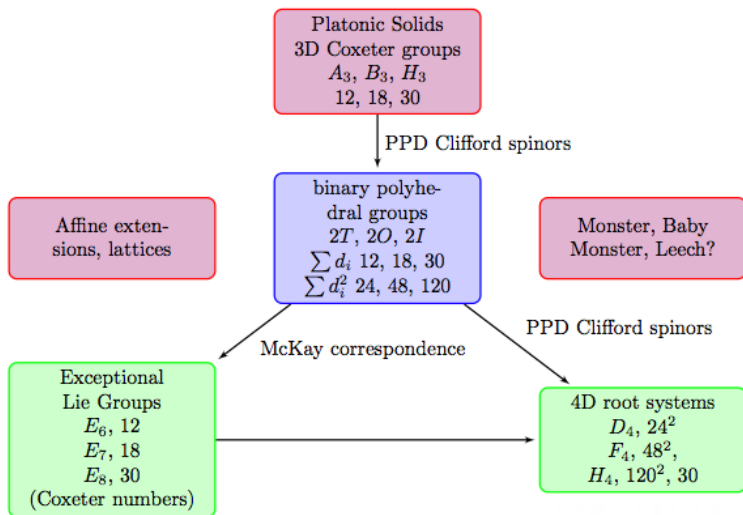
binary polyhedral groups  
 $2T, 2O, 2I$   
 $\sum d_i$  12, 18, 30  
 $\sum d_i^2$  24, 48, 120

McKay correspondence

Exceptional  
Lie Groups  
 $E_6$ , 12  
 $E_7$ , 18  
 $E_8$ , 30  
(Coxeter numbers)



# The McKay Correspondence



# The McKay Correspondence

More than E-type groups: the infinite family of 2D groups, the **cyclic** and **dicyclic groups** are in correspondence with  $A_n$  and  $D_n$ , e.g. the quaternion group  $Q$  and  $D_4^+$ . So McKay correspondence not just a trinity but **ADE-classification**. We also have  $I_2(n)$  on top of the trinity ( $A_3, B_3, H_3$ )

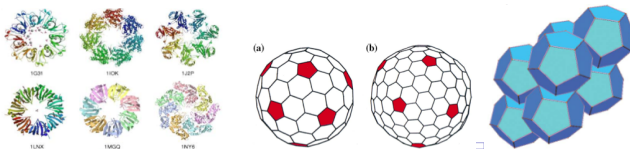
rank-3 group	diagram	binary	rank-4 group	diagram	Lie algebra	diagram
$A_1 \times A_1 \times A_1$		$Q$	$A_1 \times A_1 \times A_1 \times A_1$		$D_4^+$	
$A_3$		$2T$	$D_4$		$E_6^+$	
$B_3$		$2O$	$F_4$		$E_7^+$	
$H_3$		$2I$	$H_4$		$E_8^+$	

# 4D geometry is surprisingly important for HEP

- 4D root systems are **surprisingly relevant to HEP**
- $A_4$  is  $SU(5)$  and comes up in **Grand Unification**
- $D_4$  is  $SO(8)$  and is the little group of **String theory**
- In particular, its **triality symmetry** is crucial for showing the equivalence of RNS and GS strings
- $B_4$  is  $SO(9)$  and is the little group of **M-Theory**
- $F_4$  is the **largest crystallographic** symmetry in 4D and  $H_4$  is the **largest non-crystallographic** group
- The above are **subgroups** of the latter two
- **Spinorial nature** of the root systems could have **surprising consequences for HEP**

# Conclusions

- Novel mathematical structures – Interesting in their own right
- Numerous applications to real systems: Viruses, Proteins, Fullerenes, Quasicrystals, Tilings, Packings etc.
- Potential applications to engineering and medicine: nanotechnology and drug delivery
- Novel connection between geometry of 3D and 4D
- In fact, 3D seems more fundamental – contrary to the usual perspective of 3D subgroups of 4D groups
- Clear why spinor group gives a root system and why two factors of the same group reappear in the automorphism group



Thank you!